1. Real Analysis

An algebra of sets on X is nonempty collection A of subsets of X that is closed under finite unions and complements; in other words, if $E_1, \ldots, E_n \in A$, then $\cup_1^n E_j \in A$; and if $E \in A$, then $E^c \in A$.

There is a unique smallest σ-algebra $M(E)$ containg E,namely, the intersection of all σ-algebras containing $E. M(E)$ is called the σ-algebra generated by E.

If X is any metric space, or more generally any topological space, the σ-algebra generated by the family of open sets in X is called the Borel σ-algebra on X and is denoted by B_X.

Let X be a set equipped with a σ-algebra M. A measure on M is a function $\mu: M \to [0, \infty]$ such that

i. $\mu(\phi) = 0$,

ii. if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in M, then $\mu(\cup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$. Property (ii) is called countable additivity. It implies finite additivity:

ii'.if $E_1, \ldots, E_n$ are disjoint sets in M, then $\mu(\cup_1^n E_j) = \sum_1^n \mu(E_j)$, because one can take $E_j = \phi$ for j > n. A function μ that satisfies (i) and(ii') but not necessarily (ii) is called a finitely additive measure.

If X is a set and $M \subset P(X)$ is a σ-algebra, (X, M) is called a measurable space and the sets in M are called measurable sets. If μ is a measure on (X, M), then (X, M, μ) is called a measure space. If a statement about points $x \in X$ is true except for x in some every x. we say that it is true almost everywhere (abbreviated a.e:), or for almost every x.

A measure whose domain includes all subsets of null sets is called complete.

An outer measure on a nonempty set X is a function μ that satisfies

- $\mu^*(\phi) = 0$
- $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$.
- $\mu^*(\cup_1^\infty A_j) \leq \Sigma_1^\infty \mu^*(A_j)$.

If μ^* is an outer measure on X, a set $A \subset X$ is called μ^*-measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subset X$.

Carathéodory's Theorem.

If μ^* is an outer measure on X, the collection M of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to M is a complete measure.

A large family of measures on R whose domain is the Borel σ-algebra B_R; such measures are called Borel measures on R.

Lebesgue measure: This is the complete measure μ_F associated to the function $F(x) = x$, for which the measure of an interval is simply its length. We shall denote it by m. The domain of m is called the class of Lebesgue measurable sets, and we shall denote it by L.

The Cantor set C is the set of all $x \in [0,1]$ that have a base3 expansion $x = \Sigma a_j 3^{-j}$ with $a_j \neq 1$ for all j. Thus C is obtained from [0,1] by removing the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$, then removing the open middle thirds $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ of the two remaining intervals, and so forth.

We recall that any mapping $f: X \to Y$ between two sets induces a mapping $f^{-1}: P(Y) \to P(X)$, defined by $f^{-1}(E) = x \in X: f(x) \in E$, which preserves unions, intersections, and complements. Thus, if N is a σ-algebra on $Y, f^{-1}(E): E \in N$ is a σ-algebra on X. If (X, M) and (Y, N) are measurable spaces. a mapping $f: X \to Y$ is called (M, N)-measurable, or just measurable when M and N are understand, if $f^{-1}(E) \in M$ for all $E \in N$.

If (X, M) is a mesaurable space, a real- or complex-valued function f on X will be called M-measurable, or just measurable, if it is (M, B_R) or (M, B_C) measurable. B_R or B_C is always understood as the σ-algebra on the range space unless otherwise specified. In particular , fR $\to$ C is Lebesgue (resp. Borel) measurable if it is (L, B_C) (resp. (B_R, B_C))

measurable; likewise for $f: \mathrm{R} \to \mathrm{R}$.

The characteristic function χ_E of E is defined by $\chi * 1$ if $x \in E, * 0$ if $x \notin E$.

A simple function on X is a finite linear combination, with complex coefficients, of characteristic functions of sets in M. Equivalently, $f: X \to \mathrm{C}$ is simple iff f is measurable and the range of f is a finite subset of C. Indeed, we have $f = \Sigma_1^n z_j \chi_{E_j}$, where

$E_j = f^{-1}(z_j)$ and range $(f) = \{z_1, \dots, z_n\}$

L^+ =the space of all measurable functions from X to $[0, \infty]$. If ϕ is a simple function in L^+ with standard representation $\phi = \Sigma_1^n a_j \chi_{E_j}$, we define the integral of ϕ with respect to μ

by $\int \phi d\mu = \Sigma_1^n a_j \mu(E_j)$

The monotone convergence theorem

If f_n is a sequence in L^+such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{n\to\infty} f_n (= \sup_n f_n)$, then $\int f = \lim_{n\to\infty} \int f_n$.

Fatou's Lemma

If f_n is any sequence in L^+, then $f(\text{liminf} f_n) \leq \text{liminf} f f_n$.

If $f^+$and $f^-$are the positive and negative parts of f and at least one of $\int f^+$and $\int f^-$is finite, we define $\int f = \int f^+ - \int f^-$. We shall be mainly concerned with the case where $\int f^+$and $f f^-$ are both finite; we then say that f is integrable. Since $|f| = f^+ + f^-$, it is clear that f is integrable iff $\int |f| \leq \infty$.

If f is complex-valued measurable function, we say that f is integrable if $\int |f| \leq \infty$.

More over, if $E \in M, f$ is integrable on E if $\int_E |f| \leq \infty$.

The space of complex-valued integrable functions is a complex vector space and that the integral is a complex-linear functional on it. We denote this space-provisionally- by $L^1(\mu)($ or $L^1(X, \mu)$,, or $L^1(X)$, simply L^1, depending on the contex).

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We shall find it more convenient to redefine $L^1(\mu)$ to be the set of equivalent classes of a.e.-defined integrable functions on X, where f and g are considered equivalent iff $f = g$ a.e.

We shalll still employ the notation " $f \in L^1(\mu)$ " to mean that f is an a.e.-defined integrable function.

The dominated convergence theorem.

Let $\{f_n\}$ be a sequence in L^1 such that (a) $f_n \to f$ a.e., and (b) there exists a nonnegative $g \to L^1$ such that $|f_n| \leq g$ a.e. for all n. Then $f \to L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

Egoroff's theorem

Suppose that $\mu(X) < \infty$, and $f_1, f_2, \ldots$, and f are measurable complex-valued functions on X such that $f_n \to f$ a.e. Then for every $\epsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

The Fubini-Tonelli theorem.

Suppose that (X, M, μ) and (Y, N, ν) are σ-finite measure spaces. a.(Tonelli)If $f \in L^+(X, Y)$, then the functions $g(x) = f f_n \, \mathrm{d}\nu$ and $h(y) = \int f^y \, \mathrm{d}\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and *

$\int f \, \mathrm{d}(\mu \mathrm{x} \nu) = \int \left[\int f(x,y)\mathrm{d}\nu\right]\mathrm{d}\mu(x) = \int \left[\int f(x,y)\mathrm{d}\mu\right]\mathrm{d}\nu(y).$

b.(Fubini)If $f \in L^1(\mu \mathrm{x} \nu)$, then $f_x \in L^1(\nu)$ for a.e. x $\in X, f^y \in L^1(\mu)$. for a.e. y $\in Y$, the a.e. -defined functions $g(x) = \int f_x \, \mathrm{d}\nu$ and $h(y) = \int f^y \, \mathrm{d}\nu$ are in $L^1(\mu)$ and $L^1(\nu)$, respectively, and $(*)$ holds $dxdy = rdrd\theta$ and $dxdydz = r^2 \sin \phi dr d\theta d\phi$

$$\int_{R_n} \exp\left(-\mathrm{a}|x|^2\right)dx = \left(\frac{\pi}{a}\right)^{n/2}$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-r^2} dr = \int_{-\infty}^{\infty} e^{-r^2} dr = \sqrt{\pi}.$$

If ν is a signed measure on (X, M), a set $E \in M$ is called positive (resp.negatiive, null) for ν if $\nu(F) \geq 0$(resp. $\nu(F) \leq 0$,, $\nu(F) = 0$) for all $F \in M$ such that $F \subset E$.

The Jordan decomposition theorem

If ν is a signed measuree, there exist unique positive measures $\nu^+$and ν^-such that $\nu = \nu^+ - \nu^-$and $\nu^+ \perp \nu^-$.

The measures $\nu^+$and $\nu^-$are called the positive and negative variations of ν, and $\nu = \nu^+ - \nu^-$is called the Jordan decomposition of ν.

We define the total variation of ν to be the measure $|\nu|$ defined by $|\nu| = \nu^+ + \nu^-$.

We say that ν is absolutely continuous with respect to μ and write $\nu \ll \mu$ if $\nu(E) = 0$ for every $E \in M$ for which $\mu(E) = 0$.