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3-1 Let ν be a signed measure on (x, M). If $\{E_j\}$ is an increasing sequence in M, then $\nu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$. If $\{E_j\}$ is a decreasing sequence in M and $\nu(E_1)$ is finite, then $\nu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$.

Sol)

For the first claim, let $E_0 = \emptyset$.

Then by countable additivity, we have

$\nu(\bigcup_1^\infty E_j) = \Sigma_1^\infty \nu(E_j \setminus E_{j-1}) = \lim_{n\to\infty} \sum_1^n \nu(E_j \setminus E_{j-1}) = \lim_{n\to\infty} \nu(E_n)$

For the next claim, let $F_j = E_1 \setminus E_j$

Then $\{F_n\}$ is an increasing sequence in m

Also, $\nu(F_j) = \nu(E_1) - \nu(E_j)$, so $\nu(F_j) + \nu(E_j) = \nu(E_1)$

and $\bigcup_1^\infty F_j = \bigcup_1^\infty (E_1 \setminus E_j) = E_1 \setminus (\bigcap_1^\infty E_j)$.

Then, we can apply the previous claim, so we have

$\lim_{j\to\infty} \nu(F_j) = \nu(\bigcup_1^\infty F_j) = \nu\left(E_1 \setminus (\bigcap_1^\infty E_j)\right) = \nu(E_1) - \nu(\bigcap_1^\infty E_j)$

Therefore

$\nu(E_1) = \nu(\bigcap_1^\infty E_j) + \lim_{j\to\infty} \nu(F_j) = \nu(\bigcap_1^\infty E_j) + \lim_{j\to\infty} \left(\nu(E_1) - \nu(E_j)\right)$

Since $\nu(E_1) < \infty$, subtraction it yields

$0 = \nu\left(\bigcap_1^\infty E_j\right) - \lim_{j\to\infty} \nu(E_j)$

i.e. $\lim_{j\to\infty} \nu(E_j) = \nu(\bigcap_1^\infty E_j)$, as designed

3-2 If ν is a signed measure, E is $\nu-$null iff $|\nu|(E)$= 0. Also, if ν *and* μ are signed measures,

$\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Sol)

Suppose E is $\nu-$null. Let X=P ∪ N be a Hahn decomposition of X with respect to ν.

Since E is $\nu-$null, $\forall F \subset E$ such that F is mble, $\nu(F)$ = 0.

In particular, $\nu(E \cap P)$ = 0 and $\nu(E \cap N)$ = 0.

Thus $|\nu|(E) = \nu^+(E) + \nu^-(E)$ =$\nu(E \cap P) - \nu(E \cap N)$ =0.

Conversely, suppose $|\nu|(E)$= 0. Then $\nu^+(E) + \nu^-(E)$ = 0.

so $\nu^+(E) = \nu^-(E)$ = 0.

Now let $F \subset E$ be mable.

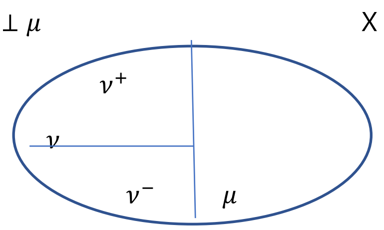
Then $\nu^+(F) \leq \nu^+(E)$ = 0, and likewise

$\nu^-(F) \leq \nu^-(E)$ = 0 so $\nu^+(F) = \nu^-(F)$ = 0. Thus $\nu(F) = \nu^+(F)$ - $\nu^-(F)$ = 0

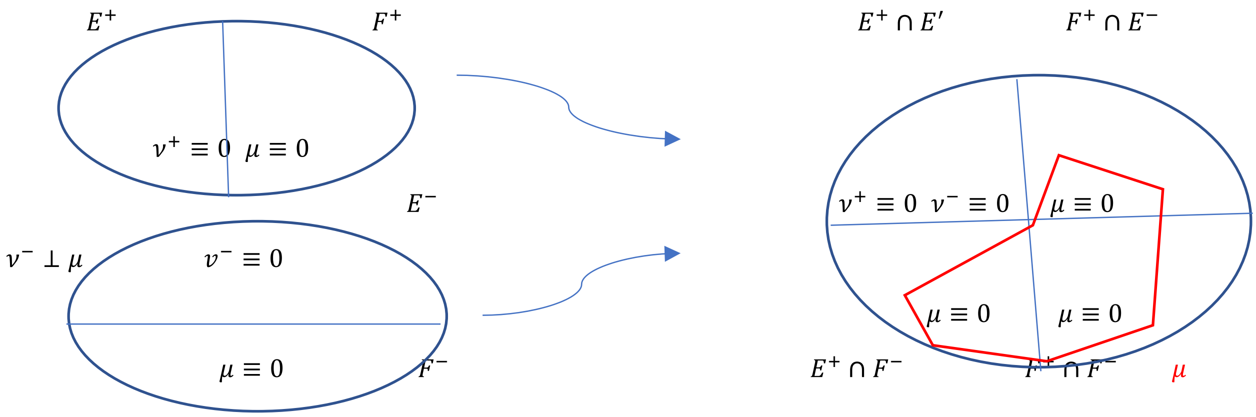
This holds $\forall F \subset E$ mable. Hence E is $\nu-$null

$\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

$\nu \perp \mu$



$\nu^+ \perp \mu$



3-3 Let ν be a signed measure on (X, M)

a. $L^1(\nu) = L^1(|\nu|)$

Sol)

Let $f \in L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$. so

$\int f\, d\nu^+ = \int_p f\, d\nu < \infty$. Likewise $\int f\, d\nu^- = -\int_N f\, d\nu > -\infty$

So $\int f\, d|\nu| = \int f\, d\nu^+ + \int f\, d\nu^- < \infty$

Hence $f \in L^1(|\nu|)$.

conversely, if $f \in L^1(|\nu|)$, then $\infty > \int f\, d|\nu| = \int f\, d\nu^+ + \int f\, d\nu^-$.

so $\int f\, d\nu^+$, $\int f\, d\nu^- < \infty$.

so $L^1(\nu^+) \cap L^1(\nu^-) = L^1(\nu)$ as desired.

b. If f∈ $L^1(\nu)$, $\left|\int f\, d\nu\right| \leq \int |f| d|\nu|$

Sol)

Let $f \in L^1(\nu)$. then

$$\begin{aligned}\left|\int f\, d\nu\right| = \left|\int_P f\, d\nu + \int_N f\, d\nu\right| &= \left|\int_P f\, d\nu^+ - \int_N f\, d\nu^-\right| \\ &\leq \left|\int_P f\, d\nu^+\right| + \left|\int_N f\, d\nu^-\right| \\ &\leq \int_P |f|\, d\nu^+ + \int_N |f|\, d\nu^- \\ &= \int |f|\, d|\nu|\end{aligned}$$

c. If $E \in M$, $|\nu|(E) = sup\left\{\left|\int_E f\, d\nu\right| : |f| \le 1\right\}$

Sol)

We have $\left|\int_E f\, d\nu\right| \le \int_E |f|\, d|\nu| \le \int_E d(|\nu|) = |\nu|(E)$

Taking the supremum over alll such f yields

$sup\left\{\left|\int_E f\, d\nu\right| : |f| \le 1\right\} \le |\nu|(E).$

Conversely, let $g = 1_P$ - 1_N. then $|g| \le 1$

and $|\nu|(E)$ = $\int_E d(|\nu|)$ = $\int_E d\nu^+ + \int_E d\nu^-$

$= \int_{E\cap p} d\nu^+ + \int_{E\cap N} d\nu^-$

$= \int_{E\cap P} g\, d\nu - \int_{E\cap N} g\, d\nu^-$

$= \int_E g\, d\nu \le \left| \int_E g\, d\nu \right|$

$\le sup\left\{\left|\int_E f\, d\nu\right| : |f| \le 1\right\}$

Hence $|\nu|(E) = sup\left\{\left|\int_E f\, d\nu\right| : |f| \le 1\right\}$

용어정리-MEASURE THEORY AND INTEGRATION- BARRA 중에서

$l(I)$ for the length of I, namely b-a

Lebesgur outer measure(outer measure) $m^*(A) = \inf \Sigma\, l(I_n)$

Where the infimum is taken over all finite or countable collections of intervals $[I_n]$ such that $A \subseteq I_n$

The set E is Lebegue measurable(measurable) if for each set A we have

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

As m^* is subadditive, to prove E is measurable we need only show, for each A, that

$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$

A class of subsets of an arbitrary space X is said to be a σ-algebra if X belongs to the class and the class is closed under the formation of countable unions and of complements

only finite unions we obtain an algebra

M : the class of Lebegue measurable sets

Let A be a class of subsets of a space X. Then there exists a smallest σ-algebra S containing A.

We say that S is the σ-algebra generated by A

We denoted by B the σ-algebra generated by the class of intervals of the form [a,b); its members are called the Borel sets of R **

$lim\,sup\, E_i = \bigcap_1^\infty \bigcup_{i \geq n} E_i$ $lim\,inf\, E_i = \bigcup_1^\infty \bigcap_{i \geq n} E_i$

if $E_1 \subseteq E_2 \subseteq \ldots$, we have $m(lim\, E_i) = lim\, m(E_i)$

if $E_1 \supseteq E_2 \supseteq \ldots$, and $m(E_i) < \infty$ for each I, then we have $m(lim\, E_i) = lim\, m(E_i)$

Let f be an extended real-valued function defined on a measurable set E

Then f is a Lebesgue-mesurable function (measurable function) if, for each $\alpha \in R$, the set

$[x : f(x) > \alpha]$ is measurable

** we say that the function f is Borel measurable or a Borel function if $\forall \alpha$, $[x : f(x) > \alpha]$ is a Borel set

Let E be a measurable set. Then for each y the set $E + y$ =[$x + y$: $x \in E$] is measurable and the measures are the same.

A non-negative finite-valued function $\varphi(x)$, taking only a finite number of different values, is called a simple function. If $a_1, a_{2,} \dots, a_n$ are the distint values taken by φ and $A_i = [x: \varphi(x) = a_i]$, then clearly $\varphi(x) = \sum_1^n a_i \chi_{A_i}(x)$

The sets A_i are measurable if φ is a measurable funtion

Let φ be a measurable simple function. Then $\int \varphi\, dx = \sum_1^n a_i m(A_i)$

where $a_{i\,,} A_i$, i=1,...,n are as in $\varphi(x) = \sum_1^n a_i \chi_{A_i}(x)$ is called the integral of φ

For any non-negative measurable function f, the integral of f, $\int f\, dx$, is given by

$\int f\, dx$ =sup $\int \varphi\, dx$, where the supremum is taken over all measurable simple funtions $\varphi, \varphi \leq f$.

$$\int_E \varphi\, dx = \sum_1^n a_i m(A_j \cap E)$$

$$\int_{A \cup B} \varphi\, dx = \int_A \varphi\, dx + \int_B \varphi\, dx$$

Lebsgue's Monotone Convergence Theorem

Let { f_n, n=1,2,...} be a sequence of non-negative measurable functions such that $\{f_n\}$ is monotone increasing for each x. Let $f = lim\, f_n$. Then $\int f\, dx = lim \int f_n\, dx$.

Let f and g be non-negative measurable functions. Then $\int f\, dx + \int g\, dx = \int (f + g)\, dx$

If $f(x)$ is any real fuction, $f^+ = max(f(x), 0)$, $f^-(x) = max(-f(x), 0)$, are said to be the positive and negative parts of f, respectively

$f = f^+ - f^-$

$|f| = f^+ + f^{-1}$

$f^+, f^- \geq 0$

f is measurable iff f^+ and f^{-1} are both measurable

If f is a measurable function and $\int f^+\, dx < \infty$, $\int f^-\, dx < \infty$, we say that f is integrable and its integrable is given by $\int f\, dx = \int f^+\, dx - \int f^-\, dx$.

$* \int |f|\, dx = \int f^+\, dx + \int f^-\, dx$

If E is a measurable set, f is a measurable function, and $\chi_E f$ is integrable, we say that f is integrable over E, and its integral is given by $\int_E f\, dx = \int f \chi_E\, dx$. The notation $f \in L(E)$ is then sometimes used.

3-4 If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^{+}$ and $\mu \geq \nu^{-}$.

Sol)

a)

Let $P \cup N$ be a Hahn decomposition for ν.

Let $E \in M$, We want to show $\lambda(E) \geq \nu^{+}(E)$, i.e.

$\lambda(E \cap P) + \lambda(E \cap N) = \lambda(E) \geq \nu^{+}(E)$

$=(\lambda - \mu)(P \cap E)$

$= \lambda(P \cap E) - \mu(P \cap E)$

So we want to show $\lambda(E \cap N) \geq -\mu(P \cap E)$

This is trvial since $\mu,\ \lambda \geq 0$.

b)

Let $P \cup N$ be a Hahn decomposition for ν.

Let $E \in M$, We want to show $\mu(E) \geq \nu^{-}(E)$. i.e.

$\mu(E \cap P) + \mu(E \cap N) = \mu(E) \geq \nu^{-}(E)$

$=-(\lambda - \mu)(E \cap N)$

$=-\lambda(E \cap N) + \mu(E \cap N)$

Hence we want to show $\mu(E \cap P) \geq -\lambda(E \cap N)$, which is trivial.

3-5 If ν_1, ν_2 are signed measures that both omit the value $+\infty$ or $-\infty$, then $|\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|$.

Sol)

Since ν_1, ν_2 both omit either $+\infty$ or $-\infty$,

so we can write $\nu_1 + \nu_2 = (\nu_1^+ - \nu_1^-) + (\nu_2^+ - \nu_2^-)$

$$= (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)$$

$$=: \lambda - \mu$$

By exercise4,

$\lambda \ge (\nu_1 + \nu_2)^+$ and $\mu \ge (\nu_1 + \nu_2)^-$. so

$$|\nu_1| + |\nu_2| = \nu_1^+ + \nu_1^- + \nu_2^+ + \nu_2^- = \lambda - \mu$$

$$\ge (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^-$$

$$= |\nu_1 + \nu_2|$$

3-6 Suppose $\nu(E) = \int f\, du$ where μ is a positive measure and f is an extended μ-integrable function. Describe the Hahn decomposition of ν and the positive, negative, and total variations of ν in terms of f and μ.

Sol)

I claim P={$f \geq 0$}, N={$f < 0$}, $\nu^+ = f^+\, d\mu$, $\nu^- = f^-\, d\mu$, and $|\nu| = |f|\, d\mu$.

WLOG, assume $\int f^-\, d\mu < \infty$.

Now $P \cup N$=X and $P \cap N$ =∅.

Also, $E \subset P$ => $\nu(E \cap P)$ = $\int_{E\cap p} f\, d\mu$ = $\int_{E\cap p} f^+\, d\mu$ ≥ 0,

and $E \subset N$ => $\nu(E \cap N)$ = $\int_{E\cap N} f\, d\mu$ =$- \int_{E\cap N} f^-\, d\mu$ ≤ 0. so

P is positive set and N is a negative set. Hence P∪N is a Haha decomposition of X.

with respect to ν. Next, $\forall E \in M$,

$\nu^+(E) = \int_{E\cap P} f\, d\mu$ = $\int_{E\cap p} f^+\, d\mu$ = $\int_E f^+\, d\mu$,

so $\nu^+ = f^+\, d\mu$.

likewise, $\forall E \in M$,

$\nu^-(E)$ = $\nu^+(E) - \nu(E)$ = $\int_E f^+\, du$ - $\int_E f\, du$ = $\int_{E\cap p} f\, du$ -$\int_E f\, du$ = $\int_{E\cap N} -f\, du$ = $\int_E f^-\, d\mu$

so $\nu^- = f^-\, d\mu$.

Furthermore, , $\forall E \in M$,

$|\nu|(E)$ = $\nu^+(E) + \nu^-(E)$ = $\int_E f^+\, d\mu$ + $\int_E f^-\, d\mu$ = $\int_E f^+ + f^-\, d\mu$ = $\int_E |f|\, d\mu$.

so $|\nu| = |f|\, d\mu$

3-7 Suppose that ν is a signed measure on (X, M) and $E \in M$.

a. $\nu^+(E)$ = sup { $\nu(F)$: $F \in M,\ F \subset E$} and $\nu^-(E)$ = -inf{ $\nu(F)$: $F \in M,\ F \subset E$ }

Sol)

Let $X = P \cup N$. Let $E \epsilon M$. Then $\nu^+(E) = \nu(E \cap P) \le$ sup { $\nu(F)$: $F \in M,\ F \subset E$}.

Also, if $F \subset E$, then $F \cap P \subset E \cap P$,

so $\nu(F) = \nu(F \cap P) + \nu(F \cap N) \le \nu(F \cap P) = \nu^+(F) \le \nu^+(E)$.

Taking the supremum ove all such F yeilds

sup { $\nu(F)$: $F \subset E$ } $\le \nu^+(E)$.

Hence = holds.

As for ν^-, we have $\nu^-(E) = -\nu(E \cap N) \le$ -inf { $\nu(F)$: $F \subset E$ }.

Next, if $F \subset E$, then $F \cap N \subset E \cap N$, so

$\nu(F) = \nu(F \cap P) + \nu(F \cap N) \ge \nu(F \cap N) = -\nu^-(F) \ge -\nu^-(E)$

So $\nu^-(E) \le -\nu(F)$

So $\nu^-(E) \le$ sup { $-\nu(F)$: $F \subset E$ } = -inf { $\nu(F)$: $F \subset E$ }

Hence = holds.

b. $|\nu|(E) = \sup \{ \sum_1^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \dots, E_n$ are disjoint, and $\cup_1^n E_j = E \}$

Sol)

First, $\sup \{ \sum_1^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \dots, E_n$ are disjoint, and $\cup_1^n E_j = E\} \geq |\nu(E \cap P)| + |\nu(E \cap N)|$

$$= \nu^+(E) + |-\nu^-(E)|$$

$$= |\nu|(E)$$

Conversely, let $E = \bigcup_1^n E_j$.

Then $|\nu|(E) = |\nu|(\bigcup_1^n E_j) = \sum_1^n |\nu|(E_j) = \sum_1^n \left(\nu^+(E_j) + \nu^-(E_j)\right)$

$$\geq \sum_1^n \left(\nu^+(E_j) - \nu^-(E_j)\right)$$

$$= \sum_1^n |\nu(E_j)|.$$

so taking supremum over all such $(E_j)_1^n$ yields

$|\nu|(E) \geq \sup \{ \sum_1^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \dots, E_n$ are disjoint, and $\cup_1^n E_j = E \}$

Thus = holds.

용어 정리2- 뒤에서 앞으로

The Lebegue Radon Nicodym Theorem

Let ν be a σ finite signed measure and μ a σ finite positive measure on (X, M)

There exist unique **σ finite signed measure** $\lambda \perp \mu$, $\rho \ll \mu$, and $\nu = \lambda + \rho$.

Moreover, there is **an extended μ integrable function f**$: X \to \mathbb{R}$ such that $d\rho = f\, d\mu$,

and any two such functions are equal $\mu\, a.e.$

Theorem

Let ν be a finite measure and μ a positive measure on (X, M).

Then $\nu \ll \mu$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Corollary

If $f \in L^1(\mu)$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|\int_E f\, d\mu\right| < \varepsilon$ whenever $\mu(E) < \delta$.

ν is a signed measure and μ is a positive measure on (X, M).

We say that ν is absolutely continuous with respect to μ and write $\nu \ll \mu$

if $\nu(E) = 0$ for every $E \in M$ for which $\mu(E) = 0$

It is easily verified that $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Integration with respect to a signed measure ν is defined in the obvious way: We set

$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$

$\int f\, d\nu = \int f\, d\nu^+ - \int f\, d\nu^-$ ($f \epsilon L^1(\nu)$)

The Jordan Decomposition Theorem

If ν is a signed measure, there exist unique positive measures ν^+ and ν^- such that

$\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$

ν^+ positive variation of ν

ν^- negative variation of ν

$\nu = \nu^+ - \nu^-$ Jordan decomposition of ν

$|\nu| = \nu^+ + \nu^-$ total variation of ν

ν null iff $|\nu|(E) = 0$, and $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$

If ν is a signed measure on (X, M), a set $E \in M$ is called

positive for ν if $\nu(F) \geq 0$

negative for ν if $\nu(F) \leq 0$

null for ν if $\nu(F) = 0$ for all $F \in M$ such that $F \subset E$

Thus, in the example $\nu(E) = \int_E f\, d\mu$ described above,

E is positive when $f \geq 0$

negative when $f \leq 0$

or null precisely when $f = 0$ μ a.e. on E

First, if μ_1, μ_2 are measures on M and at least one of them is finite, then $\nu = \mu_1 - \mu_2$ is a signed measure.

Second, if μ is a measure on M and $f : X \to [-\infty, \infty]$ is a measurable fuction such that at least one of $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ is finite

we shall call f an extended μ integrable function

the set function ν defined by $\nu(E) = \int_E f \, d\mu$

Let (X, M) be a measurable space. A signed measure on (X, M) is a function

$\nu: M \to [-\infty, \infty]$ such that

$\nu(\phi) = 0$

ν assumes at most one of the values $\pm\infty$

if $\{E_j\}$ is a sequence of disjoint sets in M, then $\nu(\bigcup_1^\infty E_j) = \sum_i^\infty \nu(E_j)$,

where the latter sum converges absolutely if $\nu(\bigcup_1^\infty E_j)$ is finite

Thus every measure is a signed measure

we shall sometimes refer to measures as positive measures.

$|\nu|(E) = \nu^+(E) + \nu^-(E)$

$\nu^+(E) = \int_E f^+ \, d\mu$

$\nu^-(E) = \int_E f^- \, d\mu$

$f^+ + f^- = |f|$

$\int_P f \, d\nu = \int_P f \, d\nu^+$

$$\int_N f \, d\nu = -\int_N f \, d\nu^-$$

$|\nu|(E) = \nu^+(E) + \nu^-(E) = \int f \, d\,|\nu| = \int f \, d\nu^+ + \int f \, d\nu^-$

a. $|\nu|(E) = \nu^+(E) + \nu^-(E) = \int f \, d\,|\nu| = \int f \, d\nu^+ + \int f \, d\nu^-$

b. $\nu = \nu^+ - \nu^- = \int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-$

Radon-Nikodym theorem

The Radon Nikodym theorem involves a measurable space (X, Σ) on which two σ finite measurea are defined, μ and ν. It states that, if $\nu \ll u$, then there exists a Σ measurable function

$f: X \to [0, \infty)$, such that for any measurable set $A \subset X$,

$\nu(A) = \int_A f \, d\mu$.

The function f satisfying the above equality is uniquely defined up to a μ null set, that is, if g is another function which the same property, then $f = g$ μ almost everywhere.

Extension to signed measure

A similar theorem can be proven for signed measure; namely, that if μ is a nonnegative σ finite measure, and ν is a finite valued signed measure such that $\nu \ll u$, that is ν is absolutely continuous with respect to μ, then there ia a μ integrable real valued function g on X such that for every measurable set A,

$\nu(A) = \int_A g \, d\mu$.

용어정리3

Measure Mesurable function **Generated σ-algebra** Measurable space **Measure space**

Signed measure **σ-algebra** algebra

Let X be a set equipped with a σ-algebra M.

A measure on M is a function μ : $M \rightarrow [0,\infty]$ such that

- $\mu(\phi)$ =0 (1)
- if $\{E_j\}_1^\infty$ is sequence of disjoint sets in M, then $\mu(\bigcup_1^\infty E_j) = \Sigma_1^\infty \mu(E_j)$ (2)

(2) is called countable additivity

It implies finite additivity

- if $E_1, \ldots E_n$ are disjoint sets in M, then $\mu(\bigcup_1^n E_j) = \Sigma_1^n \mu(E_j)$ (3)

because one can take $E_j = \phi$ for $j > n$.

A function μ that satisfies (1) and (3) but not necessarily (2) is called a finitely additive measure

If X is a set and $M \subset P(X)$ is a σ-algebra, (X, M) is called a measurable space

and the sets in M are called measurable sets

If μ is a measure on (X, M), then (X, M, μ) is called a measure space

We recall that any mapping $f: X \rightarrow Y$ between two sets induces a mapping

$f^{-1}: P(Y) \rightarrow P(X)$, defined by $f^{-1}(E)$ = { $x \in X$: $f(x) \in E$ },

which preserves unions, intersections, and complements.

If (X, M) and (Y, N) are measurable spaces, a mapping $f: X \rightarrow Y$ is called (M, N)-measureable, or

just measurable when M and N are understood, if $f^{-1}(E) \in M$ for all $E \in N$

We now examine the most important measure on $\mathbb{R}$, namely, Lebegue measure:

This is the complete measure μ_F associated to the function $F(x) = x$,

for which m is called the class of Lebegue measurable sets, and shall denoted it by $\mathcal{L}$

We shall also refer to the restriction of m to $B_{\mathbb{R}}$ as Lebegue measure

Our first applications of Caratheodory's theorem will be in the context of extending measures from algebras to σ-algebras. More precisely, if $A \subset P(X)$ is an algebra, a function $\mu_0 : A \rightarrow [0,\infty]$ will be called a premeasure if

- $\mu_0(\phi) = 0$
- if $\{A_j\}_1^\infty$ is a sequence of disjoint sets in A such that $\bigcup_1^\infty A_j \in A$, then

$\mu_0(\bigcup_1^\infty A_j) = \Sigma_1^\infty \mu_0(A_j)$

In particular, a premeasure is finitely additive since one can take $A_j = \phi$ for j large.

The notions of finite and σ-finite premeasures are defined just as for measures

The abstract generalization of the notion of outer area is as follows.

An outer measure on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \rightarrow [0,\infty]$ that satisfies

- $\mu^*(\phi) = 0$
- $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$
- $\mu^*(\bigcup_1^\infty A_j) \leq \Sigma_1^\infty \mu^*(A_j)$

Let (X, M) be a measurable space.

A signed measure on (X, M) is a function $v : M \rightarrow$ [-∞,∞] such that

- $v(\phi)$ = 0
- v assumes at most one of the values $\pm$ ∞
- if $\{E_j\}$ is a sequence of disjoint sets in M, then $v(\bigcup_1^\infty E_j)$ = $\Sigma_1^\infty v(E_j)$

where the latter sum converges absolutely if $v(\bigcup_1^\infty E_j)$ is finite

The most common way to obtain outer measure is to start with a family $\mathcal{E}$ of "elementary sets" on which a notion of measure of defined and then to approximate arbitrary sets "from the outside" by countable unions of members of $\mathcal{E}$

The measure $\bar{\mu}$ is called the completion of μ, and $\bar{M}$ is called the completion of M with respect to μ

If (x, M, μ) is a mesure space, a set $E \in M$ such that $\mu(E)$=0 is called a null set

By subadditivity, any countable union of null sets is a null set, a fact which we shall use frequently

If a statement about points $x \in X$ is true except for x in some null set. we say that it is true almost everywhere (abbreviated a.e.) or for almost every x.

(If more precision is needed, we shall speak of a μ-null set, or μ-almost everywhere)

If $\mu(E)$=0 and $F \subset E$, then $\mu(F)$ = 0 by monotonicity provided that $F \in M$, but in general it need not be true that $F \in M$. A measure whose domain includes all subsets of null sets is called complete.

Let X be a nonempty set. An algebra of sets on X is a nonempty collection A of subsets of X that is closed under finite unions and complements; in other words,

if $E_1,\ldots,E_n \in A$, then $\bigcup_1^n E_j \in A$; and if $E \in A$, then $E^C \in A$.

A σ-algebra is an algebra that is closed under countable unions.

It is trivial to verify that the intersection of any family of σ-algebras on X is againa a σ-algebra.

It follows that if $\mathcal{E}$ is any subset of $P(X)$, there is a unique smallest σ-algebra $M(\mathcal{E})$ containing $\mathcal{E}$, namely, the intersection of all σ-algebras containing $\mathcal{E}$.

(There is always at least one such, namely, $P(X)$.) $M(\mathcal{E})$ is called the σ-algebra generated by $\mathcal{E}$.

3-8 $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$

Sol)

We will prove $\nu \ll \mu \Rightarrow \nu^+ \ll \mu$ and $\nu^- \ll \mu \Rightarrow |\nu| \ll \mu \Rightarrow \nu \ll \mu$

Thoughout this problem, let X= $P \cup N$, and $E \in M$ such that $\mu(E)$=0

First, let $\nu \ll \mu$. Then given E as above, $\mu(E)$=0, so $\nu(E) = 0$.

so $\nu^+(E) = \nu(E \cap P) \leq \mu(E) = 0$

Also, $\nu^-(E) =- \nu(E \cap N) \leq -\mu(E) = 0$

So $\nu^+(E) = \nu^-(E) = 0$

and so $\nu^+ \ll \mu$, $\nu^- \ll \mu$

Next, if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$ then $\nu^+(E) = \nu^-(E) = 0$

Then $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$,

as desired finally, if $|\nu| \ll \mu$, then $\nu^+(E) + \nu^-(E) = 0$.

so $\nu^+(E) = \nu^-(E) = 0$, and so $\nu(E) = \nu^+(E) - \nu^-(E) = 0$

Hence $\nu \ll \mu$. Therefore, the statements are equivalent

3-9 Suppose $\{\nu_j\}$ is a sequence of positive measures. If $\nu_j \perp \mu$ for all j, then $\Sigma_1^\infty \nu_j \perp \mu$; and if $\nu_j \ll \mu$ for all j, then $\Sigma_1^\infty \nu_j \ll \mu$.

Sol)

For the first part, for each $j \in \mathbb{N}$ $(\not\ni 0)$

let $X = N_j \cup M_j$ where ν_j lives on N_j and μ lives on M_j

Let $\nu = \Sigma_1^\infty \nu_j$. Then ν is a measure

$\nu(\phi) = \sum_1^\infty \nu_j(\phi) = \Sigma_1^\infty 0 = 0$, and if (E_j) disjoint, $\nu(\cup E_j)$ = $\sum_{n=1}^{\infty} \nu_n\left(\bigcup_{j=1}^{\infty} E_j\right)$

= $\Sigma_{n=1}^{\infty}\Sigma_{j=1}^{\infty}\nu_n(E_j)$

= $\Sigma_{j=1}^{\infty}\Sigma_{n=1}^{\infty}\nu_n(E_j)$

= $\Sigma_{j=1}^{\infty}\nu(E_j)$

Now that we know ν is a measure,

Let $N = \bigcup_1^{\infty} N_j$ and $M = \bigcap_1^{\infty} M_j$

I claim μ lives on M, ν on N

Let $E \subset N$. then $\forall j \in \mathbb{N}$, $\mu(E \cap N_j)$ = 0.

So $\mu(E)$ $\leq \mu(\cup_1^{\infty} E \cap N_j)$

= $\Sigma_1^{\infty}\mu(E \cap N_j)$

= $\Sigma_1^{\infty}0$

= 0

Thus N is null for μ. Also $\forall E \subset M$, $E \subset M$

$\forall j \in \mathbb{N}$, so $\nu_j(E)$ = 0.

Thus $\nu(E)$ = $\Sigma_1^{\infty}\nu_j(E)$ = $\Sigma_1^{\infty}0$ = 0

So M is null for ν

Also, $N \cup M$ = X. Hence $\nu \perp \mu$

For the second part, Suppose $\nu_j \ll \mu$ $\forall j \in \mathbb{N}$

Then $\nu_j(E)$ = 0 $\forall j$

So $\nu(E)$ = $\sum_1^{\infty} \nu_n(En)$ = 0

3-10 Theorem 3.5 may fail when ν is not finite.(Consider $d\nu(x) = {dx}/{x}$ and $d\mu(x) = dx$ on (0,1), or ν = counting measure and $\mu(E) = \sum_{n\in E} 2^{-n}$ on $\mathbb{N}$.)

Sol)