용어정리 1 Real Analysis – Folland 책 중에서

Measure Mesurable function **Generated σ-algebra** Measurable space **Measure space**

Signed measure **σ-algebra** algebra

Let X be a set equipped with a σ-algebra M.

A measure on M is a function μ : M → [0,∞] such that

- $\mu(\phi)$ =0 (1)
- if $\{E_j\}_1^\infty$ is sequence of disjoint sets in M, then $\mu(\bigcup_1^\infty E_j) = \Sigma_1^\infty \mu(E_j)$ (2)

(2) is called countable additivity

It implies finite additivity

- if $E_1, \dots E_n$ are disjoint sets in M, then $\mu(\bigcup_1^n E_j) = \Sigma_1^n \mu(E_j)$ (3)

because one can take $E_j = \phi$ for $j > n$.

A function μ that satisfies (1) and (3) but not necessarily (2) is called a finitely additive measure

If X is a set and $M \subset P(X)$ is a σ-algebra, (X, M) is called a measurable space

and the sets in M are called measurable sets

If μ is a measure on (X, M), then (X, M, μ) is called a measure space

We recall that any mapping $f: X \to Y$ between two sets induces a mapping

$f^{-1}: P(Y) \to P(X)$, defined by $f^{-1}(E) = \{ x \in X : f(x) \in E \}$,

which preserves unions, intersections, and complements.

If (X, M) and (Y, N) are measurable spaces, a mapping $f: X \to Y$ is called (M, N)-measureable, or

just measurable when M and N are understood, if $f^{-1}(E) \in M$ for all $E \in N$

We now examine the most important measure on $\mathbb{R}$, namely, Lebegue measure:

This is the complete measure μ_F associated to the function $F(x) = x$,

for which m is called the class of Lebegue measurable sets, and shall denoted it by $\mathcal{L}$

We shall also refer to the restriction of m to $B_{\mathbb{R}}$ as Lebegue measure

Our first applications of Caratheodory's theorem will be in the context of extending measures from algebras to σ-algebras. More precisely, if $A \subset P(X)$ is an algebra, a function $\mu_0 : A \to [0,\infty]$ will be called a premeasure if

- $\mu_0(\phi) = 0$
- if $\{A_j\}_1^\infty$ is a sequence of disjoint sets in A such that $\bigcup_1^\infty A_j \in A$, then

$\mu_0(\bigcup_1^\infty A_j) = \Sigma_1^\infty \mu_0(A_j)$

In particular, a premeasure is finitely additive since one can take $A_j = \phi$ for j large.

The notions of finite and σ-finite premeasures are defined just as for measures

The abstract generalization of the notion of outer area is as follows.

An outer measure on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ that satisfies

- $\mu^*(\phi)=0$
- $\mu^*(A) \le \mu^*(B)$ if $A \subset B$
- $\mu^*(\bigcup_1^\infty A_j) \le \Sigma_1^\infty \mu^*(A_j)$

Let (X, M) be a measurable space.

A signed measure on (X, M) is a function $v : M \to [-\infty,\infty]$ such that

- $v(\phi) = 0$
- v assumes at most one of the values $\pm\infty$
- if $\{E_j\}$ is a sequence of disjoint sets in M, then $v(\bigcup_1^\infty E_j) = \Sigma_1^\infty v(E_j)$

where the latter sum converges absolutely if $v(\bigcup_1^\infty E_j)$ is finite

The most common way to obtain outer measure is to start with a family $\mathcal{E}$ of "elementary sets" on which a notion of measure of defined and then to approximate arbitrary sets "from the outside" by countable unions of members of $\mathcal{E}$

The measure $\bar{\mu}$ is called the completion of μ, and $\bar{M}$ is called the completion of M with respect to μ

If (x,M,μ) is a mesure space, a set $E \in M$ such that $\mu(E)=0$ is called a null set

By subadditivity, any countable union of null sets is a null set, a fact which we shall use frequently

If a statement about points $x \in X$ is true except for x in some null set. we say that it is true almost everywhere (abbreviated a.e.) or for almost every x.

(If more precision is needed, we shall speak of a μ-null set, or μ-almost everywhere)

If $\mu(E)=0$ and $F \subset E$, then $\mu(F) = 0$ by monotonicity provided that $F \in M$, but in general it need not be true that $F \in M$. A measure whose domain includes all subsets of null sets is called complete.

Let X be a nonempty set. An algebra of sets on X is a nonempty collection A of subsets of X that is closed under finite unions and complements; in other words,

if $E_1,\ldots,E_n \in A$, then $\bigcup_1^n E_j \in A$; and if $E \in A$, then $E^C \in A$.

A σ-algebra is an algebra that is closed under countable unions.

It is trivial to verify that the intersection of any family of σ-algebras on X is againa a σ-algebra.

It follows that if $\mathcal{E}$ is any subset of $P(X)$, there is a unique smallest σ-algebra $M(\mathcal{E})$ containing $\mathcal{E}$, namely, the intersection of all σ-algebras containing $\mathcal{E}$.

(There is always at least one such, namely, $P(X)$.) $M(\mathcal{E})$ is called the σ-algebra generated by $\mathcal{E}$.

용어정리 2-MEASURE THEORY AND INTEGRATION- BARRA 중에서

$l(I)$ for the length of I, namely b-a

Lebesgur outer measure(outer measure) $m^*(A) =$ inf $\Sigma\, l(I_n)$

Where the infimum is taken over all finite or countable collections of intervals $[I_n]$ such that $A \subseteq I_n$

The set E is Lebegue measurable(measurable) if for each set A we have

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

As m^* is subadditive, to prove E is measurable we need only show, for each A, that

$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$

A class of subsets of an arbitrary space X is said to be a σ-algebra if X belongs to the class and the class is closed under the formation of countable unions and of complements

only finite unions we obtain an algebra

M : the class of Lebegue measurable sets

Let A be a class of subsets of a space X. Then there exists a smallest σ-algebra S containing A.

We say that S is the σ-algebra generated by A

We denoted by B the σ-algebra generated by the class of intervals of the form [a,b); its members are called the Borel sets of R **

$lim\,sup\, E_i = \bigcap_1^{\infty} \bigcup_{i \geq n} E_i$ $lim\,inf\, E_i = \bigcup_1^{\infty} \bigcap_{i \geq n} E_i$

if $E_1 \subseteq E_2 \subseteq \ldots$, we have $m(lim\, E_i) = lim\, m(E_i)$

if $E_1 \supseteq E_2 \supseteq \ldots$, and $m(E_i) < \infty$ for each I, then we have $m(lim\, E_i) = lim\, m(E_i)$

Let f be an extended real-valued function defined on a measurable set E

Then f is a Lebesgue-mesurable function (measurable function) if, for each $\alpha \in R$, the set

$[x: f(x) > \alpha]$ is measurable

** we say that the function f is Borel measurable or a Borel function if $\forall \alpha$, $[x: f(x) > \alpha]$ is a Borel set

Let E be a measurable set. Then for each y the set $E + y$ =[$x + y$: $x \in E$] is measurable and the measures are the same.

A non-negative finite-valued function $\varphi(x)$, taking only a finite number of different values, is called a simple function. If $a_1, a_{2,} \dots, a_n$ are the distint values taken by φ and $A_i = [x: \varphi(x) = a_i]$, then clearly $\varphi(x) = \sum_1^n a_i \chi_{A_i}(x)$

The sets A_i are measurable if φ is a measurable funtion

Let φ be a measurable simple function. Then $\int \varphi\, dx = \sum_1^n a_i m(A_i)$

where $a_{i}, A_i,$ i=1,...,n are as in $\varphi(x) = \sum_1^n a_i \chi_{A_i}(x)$ is called the integral of φ

For any non-negative measurable function f, the integral of f, $\int f\, dx$, is given by

$\int f\, dx$ =sup $\int \varphi\, dx$, where the supremum is taken over all measurable simple funtions $\varphi, \varphi \leq f$.

$$\int_E \varphi\, dx = \sum_1^n a_i m(A_j \cap E)$$

$$\int_{A \cup B} \varphi\, dx = \int_A \varphi\, dx + \int_B \varphi\, dx$$

Lebsgue's Monotone Convergence Theorem

Let { f_n, n=1,2,...} be a sequence of non-negative measurable functions such that $\{f_n\}$ is monotone increasing for each x. Let $f = lim\, f_n$. Then $\int f\, dx = lim \int f_n\, dx$.

Let f and g be non-negative measurable functions. Then $\int f\, dx + \int g\, dx = \int (f+g)\, dx$

If $f(x)$ is any real fuction, $f^+ = max(f(x), 0)$, $f^-(x) = max(-f(x), 0)$, are said to be the positive and negative parts of f, respectively

$f = f^+ - f^-$

$|f| = f^+ + f^{-1}$

$f^+, f^- \geq 0$

f is measurable iff f^+ and f^{-1} are both measurable

If f is a measurable function and $\int f^+\, dx < \infty$, $\int f^-\, dx < \infty$, we say that f is integrable and its integrable is given by $\int f\, dx = \int f^+\, dx - \int f^-\, dx$.

$*$ $\int |f|\, dx = \int f^+\, dx + \int f^-\, dx$

If E is a measurable set, f is a measurable function, and $\chi_E f$ is integrable, we say that f is integrable over E, and its integral is given by $\int_E f\, dx = \int f \chi_E\, dx$. The notation $f \in L(E)$ is then sometimes used.

용어 정리 3- 뒤에서 앞으로

The Lebegue Radon Nicodym Theorem

Let ν be a σ finite signed measure and μ a σ finite positive measure on (X, M)

There exist unique **σ finite signed measure** $\lambda \perp \mu$, $\rho \ll \mu$, and $\nu = \lambda + \rho$.

Moreover, there is **an extended μ integrable function f**$: X \to \mathbb{R}$ such that $d\rho = f\,d\mu$,

and any two such functions are equal $\mu\, a.e.$

Theorem

Let ν be a finite measure and μ a positive measure on (X, M).

Then $\nu \ll \mu$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Corollary

If $f \in L^1(\mu)$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|\int_E f\,d\mu\right| < \varepsilon$ whenever $\mu(E) < \delta$.

ν is a signed measure and μ is a positive measure on (X, M).

We say that ν is absolutely continuous with respect to μ and write $\nu \ll \mu$

if $\nu(E) = 0$ for every $E \in M$ for which $\mu(E) = 0$

It is easily verified that $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Integration with respect to a signed measure ν is defined in the obvious way: We set

$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$

$\int f\,d\nu = \int f\,d\nu^+ - \int f\,d\nu^-$ ($f \epsilon L^1(\nu)$)

The Jordan Decomposition Theorem

If ν is a signed measure, there exist unique positive measures ν^+ and ν^- such that

$\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$

ν^+ positive variation of ν

ν^- negative variation of ν

$\nu = \nu^+ - \nu^-$ Jordan decomposition of ν

$|\nu| = \nu^+ + \nu^-$ total variation of ν

ν null iff $|\nu|(E) = 0$, and $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$

If ν is a signed measure on (X, M), a set $E \in M$ is called

positive for ν if $\nu(F) \geq 0$

negative for ν if $\nu(F) \leq 0$

null for ν if $\nu(F) = 0$ for all $F \in M$ such that $F \subset E$

Thus, in the example $\nu(E) = \int_E f \, d\mu$ described above,

E is positive when $f \geq 0$

negative when $f \leq 0$

or null precisely when $f = 0$ μ a.e. on E

First, if μ_1, μ_2 are measures on M and at least one of them is finite, then $\nu = \mu_1 - \mu_2$ is a signed measure.

Second, if μ is a measure on M and $f : X \to [-\infty, \infty]$ is a measurable fuction such that at least one of $\int f^+ \, d\mu$

and $\int f^- \, d\mu$ is finite

we shall call f an extended μ integrable function

the set function ν defined by $\nu(E) = \int_E f \, d\mu$

Let (X, M) be a measurable space. A signed measure on (X, M) is a function

$\nu: M \to [-\infty, \infty]$ such that

$\nu(\phi) = 0$

ν assumes at most one of the values $\pm\infty$

if $\{E_j\}$ is a sequence of disjoint sets in M, then $\nu(\bigcup_1^\infty E_j) = \sum_i^\infty \nu(E_j)$,

where the latter sum converges absolutely if $\nu(\bigcup_1^\infty E_j)$ is finite

Thus every measure is a signed measure

we shall sometimes refer to measures as positive measures.

$|\nu|(E) = v^+(E) + \nu^-(E)$

$v^+(E) = \int_E f^+ \, d\mu$

$\nu^-(E) = \int_E f^- \, d\mu$

$f^+ + f^- = |f|$

$\int_P f \, d\nu = \int_P f \, d\nu^+$

$\int_N f \, d\nu = -\int_N f \, d\nu^-$

$|\nu|(E) = v^+(E) + \nu^-(E) = \int f \, d\,|\nu| = \int f \, d\nu^+ + \int f \, d\nu^-$

a. $|\nu|(E) = v^+(E) + \nu^-(E) = \int f \, d\,|\nu| = \int f \, d\nu^+ + \int f \, d\nu^-$

b. $\nu = v^+ - \nu^- = \int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-$