19.2 Linear Systems: LU-Factorization, Matrix Inversion.

$Ax = b$ $A = [a_{jk}]$ $x^{\top} = [x_1 \dots x_n]$ $\mathrm{b}^{\top} = [b_1 \cdots b_n]$

LU-factorization

$\mathbb{A} = \mathbb{L}$ ($\mathbb{L}$:lower trianguar

U: upper trianguar)

$AX = LUX = lb$

(a) $Ly = b$ (b) $TX = b$

Doolittle method
$u_{1k} = a_{1k}$

$u_{jk} = a_{jk} - \sum_{s=1}^{j-1} m_{js} u_{s_k}$

$m_{j1} = \frac{a_{j1}}{u_{11}} \quad j = 2 \dots , m$
$m_{jk} = \frac{1}{u_{kk}} \quad \left(a_{jk} - \sum_{s=1}^{k-1} m_{js} u_{a_k}\right) \quad j = k + 1, \cdots , m; \; k \geqq 2$

Crout's method

$m_{j1} = a_{j1} \quad j = 1, \cdots , n$

$m_{jk} = a_{jk} - \sum_{s=1}^{k-1} m_{js} u_{sk} \qquad j = k, \cdots , m; \quad k \geq 2$

$u_{1k} = \frac{a_{1k}}{m_{11}}$

$u_{1k} = \frac{a_{1k}}{m_{11}} \; k = 2, \dots , m$

$u_{jk} = \frac{1}{m_{jj}} \left(a_{jk} - \sum_{s=1}^{j-1} m_{js} u_{s_k}\right) \quad k = j + 1, \cdots , n$; j ≥ 2

Cholesky's Method
* symmetric, positive definite.

$Q = x^{\top} \mathbb{C} \times$ (realquadratic)

if $Q > 0$ for all) $[x_1 \cdots x_m] \neq [0.0]$

$(A = A^\top, x^\top Ax > 0$ for all $x \neq 0)$

$* \ U = L^\top \left(u_{jk} = m_{kj}\right)$

$Ax = \mathrm{b} \quad A = LL^\top$

$m_{11} = \sqrt{a_{11}}$

$$\begin{bmatrix} m_{11} & 0 & 0 & \\ m_{21} & m_{22} & & 0 \\ m_{31} & m_{32} & & m_{23} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{bmatrix}$$

If A has an inverse, then A : nonsingular
If A hos no inverse, then A: singular

→ Gauss - Jordan Elimination

$Ax = bj$ (<= inverse of a nonsingular square matrix)

19.3 Linear Systems: Solution by Iteration.

direct methods

indirect methods or iterative method.

→ sparse

Gauss - Seidel Iteration

$$\mathbb{A} = \mathbb{I} + \mathbb{L} + \mathrm{U}$$
$$AX = (I + L + U)X = \mathrm{b}$$
$$\text{Taking } Lx \text{ and } Ux \Rightarrow x = b - Lx - Ux$$
$$x^{(m+1)} = b - Lx^{(m+1)} - Ux^{(m)}$$
$$x_j^{(m+1)} = -\frac{1}{a_{jj}}\left(\sum_{k=1}^{j-1} a_{jk}x_k^{(m+1)} + \sum_{k=j+1}^{n} a_{jk}x_k^{(m)} - b_j\right)$$

If $\max_j \left|x_j^{(m+1)} - x_j^{(m)}\right| < \varepsilon$ then
output $x^{(m+1)}$.

$* \ Ax = \lambda x \ \ x \neq 0 \Rightarrow$ eigenvalue
eigenvector

$\rightarrow (A - \lambda I)x = 0$

$A-\lambda I$ is singular iff det $(A-\lambda I)$:zero

$\Rightarrow$ Characteristic determinint

$\| A \| = \max \frac{\| Ax \|}{\| x \|}$

$\| A \| = \begin{matrix} \max \ ||Ax\| \\ \|x\|=1 \end{matrix}$

$c = \| A \| \quad \Rightarrow \| A\mathrm{x} \| \leqq \| A \| \| \mathrm{x} \|$

$\| AB \| \leqq \| A \| \ B \|$, thus $\|A^n\| \leqq \| A \|^n$

Condition number $\Rightarrow$ smal=>well conditioned;=>large=>ill conditioning
$K(A) = \| A \| \ ||A^{-1} \|$

$$\mathrm{b} = Ax$$

$$||\mathrm{b}|| = \| A \| \ ||\mathrm{x}||$$
$$\frac{1}{\| x \|} \leqslant \frac{\| A \|}{\| b \|}$$
$$x - \tilde{x} \quad = A^{-1}r$$
$$\| x - \tilde{x} \| = \|A^{-1}\| ||\mathrm{r} \| \leqslant \|A^{-1}\| ||\mathrm{r} \|$$

$$\frac{\| x - \tilde{x} \|}{\| x \|} \leqslant \frac{1}{\| x \|} \|A^{-1}\| \ \| \mathrm{r} \| \ \Big\| \leqslant \frac{\| A \|}{\| \mathrm{b} \|^{-1}} \Big\| A^{-1} \ \| r \|$$
if $k(A)$: small $\Rightarrow a$ small relative error

$\|x - \tilde{x}\| / \| x \|$, so that the system : well-conditoned.

Inaccuate matrix entries.

an inaccuracy δA of A

$$(A + \delta A)(x + \delta x) = 16$$
$$A\delta x + \delta A(x + \delta x) = 0$$

Multiplication by A^{-1}

(C) FANCY HOUSE PEOPLE

$\delta x = -A^{-1}\delta A(x + \delta x)$
A^{-1} and vector $\delta A(x + \delta x)$ instead of A and x
$\| \delta x \| = \|A^{-1}\delta A(x + \delta x)\| \leqslant \|A^{-1}\| \ \| \delta A(x + \delta x)||$

δA tand $x - \delta x$ insead of A and x

$\| \delta x \| \leq \|A^{-1}\| \delta A \| \| x + \delta x \|$

$*\ \|A^{-1}\| = K(A)/\| A \|$, division by $\| x + \delta X \|$

$$\frac{\| \delta x \|}{\| x \|} \approx \frac{\| \delta x \|}{\| x + \delta x \|} \leqslant \|A^{-1}\| \| \delta A \| = k(A) \frac{\| \delta A \|}{\| A \|}$$

$\rightarrow$ Well-condition, some inaccuraries $\| \delta A \|/\| A \|$

can have only a small effect on the solution.
→ ill-conditionny, if $\| \delta A \|/\| A \|$ is small, $\| \delta \times \|/\| x$ may be large

19.5 Method of Least Squares

$y = a + bx$
Sum of the squares of the distances of those points from the straight line is minimum

$$\left(\begin{array}{l} \frac{\partial q}{\partial a} = -2 \sum (y_j - a - bx_j) = 0 \\ \frac{\partial q}{\partial b} = -2 \sum y_j - a - bx_j) = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} a\eta + b \sum x_j = \sum y_j \\ a \sum x_j + b \sum x_j^2 = \sum x_j u_j \end{array} \right.$$

$*$ normdl equations
generalized from a polynomial $y = a + bx$

$p(x) = b_0 + b_1 x + \cdots + b_m x^{\mathrm{m}}$ where $m \leqslant \mathrm{n} - 1$

$q = \sum_{j=1}^{n} \left(y_j - p(x_j)\right)^2$ - depends in m+1 parameters $b_0, \cdots, b_m$

$\frac{\partial p}{\partial b0} = 0,\ \frac{\partial q}{\partial bm} = 0$

* $p(x) = b_0 + b_1 x + b_2 x^2 <=$ quadratic polynomial

($b_0 m + b_1 \sum x_j + b_2 \sum x_j{}^2 = \sum y_j$ normal equation

$$b_0 \sum x_j + b_1 \sum x_j^2 + b_2 \sum x_j^{-3} = \sum x_{ij} y_j$$

$$b_0 \sum x_j^2 + b_1 \sum x_j^2 + b_2 \sum x_j^4 = \sum x_j^2 y_j$$

Problom 19.5

(1)) normal equation in the case of a polynomial of the third deqree.
$b_0 n + b_1 \sum x_j + b_2 \sum x_j^2 + b_3 \sum x_j^3 = \sum y_j$ $b_0 n + b_1 \sum x_j + b_2 \sum x_j^2 + b_3 \sum x_j^3 = \sum y_j$
$b_0 \Sigma x_j + b_1 \sum x_j^2 + b_2 \sum x_3^3 + b_3 \sum x_i^4 = \Sigma x_i$ $b_0 \sum x_j + b_1 \sum x_j^2 + b_2 \sum x_j^3 + b_3 \sum x_j^4 = \sum x_j y_j$
$b_0 \sum x_j^2 + b_1 \sum x_j^3 + b_2 \sum x_j^4 + b_3 \sum x_j = \sum x_i^2 u_j$ $b_0 \sum x_j^3 + b_1 \sum x_j^4 + b_2 \sum x_j^5 + b_3 \sum x_j^6 = \sum x_j^3 y_i$

20 $y = b_0 e^{bx}$

$\ln\ y = \ln\ b_0 + bx\ y^* = a^* + bx\ y^* = \ln y \quad a^* = \ln\ b_0$

Chapter 13. Cemplex Integration.

13.1 Line Integral in the complex plane

$z(t) = x(t) + iy(t)$
$ex)\ z(t) = t + 3it$

$$\dot{z} = \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t}$$

complex plone
$(0 \leqslant t \leqslant 2)$

$(t) = x(t) + iy(t)$

$z_0, z_1, \cdots, z_{n-1}, z_n (= \mathbb{Z})$ where, $z_j = z(t_j)$
ξ_1 between z_0 and z_1 (that is, $\zeta_1 = z(t), t_0 \leqslant t \leqslant t_1$) $S_n = \sum_{m=1}^{n} f$
$(\zeta_m) \Delta Z_m$ where $\Delta Z_m = Z_m - Z_{m-1}$

- The limit of the sequence of complex numbers
  ⇒ Line integral.

seyuence of Cony
⇒ $\oint_c f(z)dz$

(one whose terminal point Z coincides with its initial point Z_0)

$f(z)$ is continuous
C is piecowise smooth the existance of the line integral

$$
\begin{aligned}
f(z) &= u(x,y) + iv(x,y) \\
\xi_m &= \xi_m + i\eta_m \text{ and } \Delta z_m = \Delta x_m + i\Delta y_m \\
S_n &= \sum (u+iv)(\Delta x_m + i\Delta y_m) \\
&\text{where } u = u(\xi_m, \eta_m),\ v = v(\xi_m, \eta_m) \\
\lim_{m\to\infty} S_n &= \int_c f(z)dz = \int_c udx - \int_c vdx + i\left[\int_c udy + \int_c v\mathrm{dx}\right].
\end{aligned}
$$

13.2 Two integration Methods

Integration by the use of the path.

$$
\int_c f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt \ \left(\dot{z} = \frac{dz}{dt}\right)
$$

$$
\begin{aligned}
&dx = \dot{x}dt \\
&dy = \dot{y}dt\ \dot{z} = \dot{z} + i\dot{y} \\
&\int_a^b f[z(t)]\dot{z}(t)dt = \int_a^b (u+iv)(x+i\dot{y})dt \\
&= \int_c [udx - vdy + i(udy + vdx)] \\
&\oint_0 \frac{dz}{z} = 2\pi i \ \ \text{(c the unit circle, counterwise)} \\
&z = \mathrm{coc}\ t + isint \\
&0 \leqslant t \leqslant 2\pi \\
&\dot{z} = -\sin\ t + i\cos\ t \\
&f[z(t)] = 1/z(t)
\end{aligned}
$$

If $m = -1$, we have $\rho^{m+1} = 1, \cos\ 0 = 1, \mathrm{cin}\ 0 = 0$ and thus, obtain 2π i. For integer $m \neq -1$ each of the two integers is zero

$$
\int_c (z - z_0)^m dz = \begin{cases} 2\pi i\ (m = -1) \\ 0\ (m \neq -1) \text{ and mfeger } \end{cases}
$$

$$
\int_a^b f(x)dx = F(b) - F(a)\ F'(x) = f(x)
$$

→ indefinite integration of analytic functions

$$
\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)\ F'(z) = f(z)
$$

Bound for the Absolute Value of Integrals.

$\left|\int_C f(z)dz\right| \leqslant ML$ L is the length C and

M a constant such that $|f(z)| \leqslant M$

$|S_n| = |\sum_{m=1}^{n} f(\xi_m)\Delta z_m| \leqq \sum_{m=1}^{n} |f(\xi_m)||\Delta z_m| \leqslant M\sum_{m=1}^{n} |\Delta Z_m|$

$|\Delta Z_m|$ in the length of the chord whose endpoints are

z_{m-1} and Zm

$Pb.$

C in the line

$$\int_0 \mathrm{Ln}_n\,(z+1)dz$$

48 .

$$|\mathrm{Ln}_n\,(z+1)| = \mathrm{In}\ |z+1| + i\mathrm{drg}\,(z+1)$$
$$|\mathrm{Ln}\,(z+1)|^2 = \ln\ |z+1|^2 + [\mathrm{drg}\,(z+1)]^2$$
$$\leqslant (\ln\ |3+4i+1|)^2 + (\mathrm{rrg}\,(3+4i+1)$$
$$= (\ln\ \sqrt{32})^2 + \left(\frac{\pi}{4}\right)^2$$
$$|\mathrm{Ln}\,(z+1)| \leqslant \sqrt{(\ln\ \sqrt{32})^2 + \left(\frac{\pi}{4}\right)^2}$$
$$||_C \mathrm{Ln}\,(z+1)dz\ | \leqslant M = \hbar\sqrt{(\ln\ \sqrt{32})^2 + \left(\frac{\pi}{4}\right)^2}$$
$$\int_C e^z dz$$
$$|e^z| = \left|e^{(3+4i)t}\right| \leqslant e^3$$
$$\left|\int_c f(z)dz\right| = \left|\int_c e^z dz\right| \leqslant 5e^3$$

$\int_C e^q dz$ upper bound : 5e e^2

$c_1:\ 0 \leqslant t \leqslant \frac{1}{2}\ c_2: \frac{1}{2} \leqslant t \leqslant 1$

$c_1 |e^z| = |e^{(3+4\cdot i)t}| \leqslant e^{\frac{3}{2}}$

$c_2\ |e^z| = |e^{(3+4-i)t}| \leqslant e^z$

$$\left|\int_c e^z dz\right| = \left|\int_{c_1} e^z dz + \int_{c_2} e^z dz\right|$$

$$\langle \left|\int_{c_1} e^z dz\right| + \left|\int_{c_2} e^z dz\right|$$

$$\leqslant e^{\frac{3}{2}} \cdot 5 + e^3 \cdot$$

$$= \frac{5}{2}\left(e^3 + e^{3/2}\right)$$

$$= 61.42.$$

13.3 Cauchy's integral Theorem

Theorem Cauchy's integral theorem

$\oint_c f(z)dz = 0$ If $f(z)$ is dudlytic ind simply connected dumbin D.

$$\int_R \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial E_1}{\partial y}\right) dxdy = \oint_C F_1 dx + F_2 dy$$